A Proof that Hitting Set is NP-Complete

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April 6, 2013

1 Introduction

In this work, we prove that Hitting Set, as defined in the next section, is an NP-Complete problem.

To do so, we will first show that Hitting Set belongs in NP, by engineering a deterministic polynomial time verifier for it. Then we will prove that it is an NP-Hard problem, by reducing a known NP-Hard problem, Vertex Cover, to Hitting Set in polynomial time.

2 Definition of Hitting Set

Let \( S \) be a finite set, \( C \) a collection of subsets \((C_1, C_2, \ldots, C_m)\) of \( S \) and \( K \) a positive integer. We will say that \((S, C)\) has a hitting set of size at most \( K \), iff:

\[ \exists S' \subseteq S : |S'| \leq K \text{ and } \forall \text{ subset } C_i \text{ in collection } C \exists c \in C_i : c \in S' \]

3 Definition of Vertex Cover

Let \( G \) be an undirected graph, \( V \) a set of nodes, \( E \) a set of edges defined over those nodes and \( K \) a positive integer. We will say that \( G \) has a a Vertex Cover of size at most \( K \), iff:

\[ \exists V' \subseteq V : |V'| \leq K \text{ and } \forall \text{ edge } e_i = \{u_i, v_i\} \in E, u_i \in V' \text{ or } v_i \in V' \]

This problem, as defined, is NP-Complete. [1]

4 Hitting Set belongs in NP

We will now describe \( V \), a deterministic polynomial time verifier for Hitting Set:

Given a finite set \( S \), a collection \( C \) of subsets of \( S \), a positive integer \( K \) and a finite set \( H \) as a certificate, \( V \) will confirm in polynomial time that \((S, C)\) has a hitting set of size at most \( K \).

Let \( \lambda \) be the sum of the sizes of all the subsets \( C_i \) in \( C \) and \( \delta \) the size of \( S \). Note that we can check if \( A \) is a subset of \( B \) with the following brute-force algorithm: \( \forall a \in A \) check if \( \exists b \in B : a=b \) which needs \( O(|A||B|) \) comparisons.

We can check if \( H \) is a subset of \( S \) that has at most \( K \) elements with \( O(K \cdot \delta) \) comparisons and if it contains at least one element from each subset \( C_i \) in the collection \( C \), with \( O(\lambda \cdot K) \) comparisons. We accept iff both checks are true.

These two checks are obviously equivalent to the problem’s definition, so Hitting Set has a polynomial time verifier. Therefore it belongs in NP. [2]
5 \textbf{Hitting Set is NP-Hard}

We will create a polynomial time reduction from Vertex Cover to Hitting Set, proving that since Vertex Cover is NP-Hard, Hitting Set must also be NP-Hard.

The reduction takes as input an undirected graph $G = (V, E)$, where $V$ is a set of nodes and $E$ a set of edges defined over those nodes, as well as a positive integer $K$ and outputs the set $V$, the collection $E = \{e_1, e_2, \ldots, e_n\}$ of subsets of $V$ and the positive integer $K$.

We claim the following equivalence holds:

"$G$ has a vertex cover of size at most $K$" \iff "$(V, E)$ has a hitting set of size at most $K$". Here is the proof:

"$G$ has a vertex cover of size at most $K$" \iff 

\[ \exists \ V' \subseteq V : |V'| \leq K \text{ and } \forall \text{ edge } e_i = \{u_i, v_i\} \in E, u_i \in V' \text{ or } v_i \in V' \iff \exists \ V' \subseteq V : |V'| \leq K \text{ and } \forall \text{ subset } e_i \text{ in collection } E \exists \ c \in e_i : c \in V' \iff \]

"$(V, E)$ has a Hitting Set of size at most $K$"

This reduction takes time linear to the size of the input (all it does is copy the input to the output), therefore polynomial. Also, as we showed, it is correct. Therefore, Hitting Set is at least as hard as Vertex Cover and since Vertex Cover is NP-Hard, so is Hitting Set. [2]

One might notice that this reduction was rather straightforward. This makes sense, since Vertex Cover is a special version of Hitting Set, where each subset $C_i$ in the collection $C$ has exactly two elements of $S$. Obviously, no problem can be harder than its generalization and since Vertex Cover is NP-Hard, Hitting Set (as a generalization of Vertex Cover) must also be NP-Hard.

6 \textbf{Hitting Set is NP-Complete}

In section 4, we proved that Hitting Set belongs in NP. In section 5, we proved that an NP-Hard problem (Vertex Cover) reduces to Hitting Set in polynomial time. Therefore, by definition, we conclude that Hitting Set is an NP-Complete problem. [2]

\textbf{References}
